JOURNAL OF APPROXIMATION THEORY 79, 307–310 (1994)

Note

Nondegeneracy, Relative Differentiability, and Integral Representation of Weak Markov Systems

FRIEDHELM SCHWENKER AND LAURA MARTIGNON

Department of Neural Information Processing, University of Ulm, Oberer Eselsberg, W—89069 Ulm (Donau), Germany

Communicated by Günther Nürnberger

Received February 22, 1993; accepted in revised form August 17, 1993

By constructing a weakly degenerate weak Markov system we show that differentiability of any order is not sufficient for the existence of an integral representation. An analytic weak Markov system is normalizable with a nonnegative function and an analytic (strong) Markov system.

Let M be a subset of the real line with card $M \geq n + 1$. It was proven by the first author of this paper [1][1]:

THEOREM 1. *A normalized weak Markov system* $\{1, f_1, \dots, f_n\} \subset F(M)$ *is representable, if and only if it has property* (E).

This result was slightly extended by Zalik in [2]. The following example shows that in the case of weak Markov systems even differentiability does not guarantee the existence of an integral representation. On the other hand, it is known that every Markov system $\{1, f_1, \dots, f_n\} \subset C^n(I)$, where I is an open interval, is representable.

EXAMPLE 1. Let $f_0 \equiv 1$ on $\mathbb{R}$. For each $t \in \mathbb{R}$, define f_1 and f_2 by

$$f_1(t) = \begin{cases} \exp(-1/t^2) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

[1] We use the terminology introduced in [1].

0021-9045/94 $6.00

and $f_2(t) = f_1(-t)$. For every $k \in \mathbb{N}$, we have

$$\lim_{t \to 0_+} \frac{\exp(-1/t^2)}{t^k} = 0.$$

It is easy to see that $\{1, f_1, f_2\}$ forms a weak Markov system on every interval I which contains the origin.

Applying the following theorem [1, Theorem 2] we obtain the existence of a class of not representable weak Markov systems in C^∞.

THEOREM 2. *Every representable weak Markov system* $\{1, f_1, \ldots, f_n\} \subset F(M)$ *is weakly nondegenerated.*

COROLLARY 1. *There exists a normalized weak Markov system on* $C^\infty(I)$, *which has no integral representation.*

Let us denote by $H(I)$ the space of all real analytic functions on an open interval I.

PROPOSITION 1. *If* $\{1, f_1, \ldots, f_n\} \subset H(I)$ *is a weak Markov system, it also forms a Markov system.*

Proof. The proof is clear for $n = 0$. Assume that $(t_0, \ldots, t_n) \in \Delta_{n+1}(I)$ are zeros of $f \in U_n \setminus \{0\}$. The Identity Theorem for analytic functions implies that there exist $z_i \in (t_{i-1}, t_i)$ with $f(z_i) \neq 0$ for all $i = 0, \ldots, n+1$ with $t_{-1} = \inf I = a$ and $t_{n+1} = \sup I = b$.

Then, the set $(z_0, t_0, z_1, t_1, \ldots, z_n, t_n, z_{n+1}) \in \Delta_{2n+1}(I)$ contains a strong oscillation of length $\geq n + 2$ of f. But this contradicts the fact that $\{1, f_1, \ldots, f_n\}$ is a weak Markov system.

Let us call a weak Markov system $\{f_0, \ldots, f_n\} \subset F(M)$ *normalizable*, if there exists a normalized weak Markov system $\{1, g_1, \ldots, g_n\} \subset F(M)$ with $f_i = f_0 \cdot g_i$ for every $i = 1, \ldots, n$.

Normalizable weak Tchebycheff systems were first studied by Zwick in [3] in the context of nondegeneracy. In the case of analytic weak Markov systems, which are a subset of the nondegenerated weak Markov systems, one obtains the following result (see also [3, Theorem 4.5]):

THEOREM 3. *Every weak Markov system* $\{f_0, \ldots, f_n\} \subset H(I)$ *is normalizable, and, the normalized system* $\{1, g_1, \ldots, g_n\} \subset H(I)$ *forms a Markov system.*

Proof. Denote by $N(f_0)$ the set of zeros of f_0. Clearly $N(f_0)$ is countable, and its only accumulation points are the extremes of the interval, so that we have $N(f_0) = \{t_j : j \in Z \subset \mathbb{Z}\}$. Therefore, the set $\{1, g_1, \ldots, g_n\}$ defined by $g_i = f_i / f_0$, for $i = 1, \ldots, n$ is a weak Markov system on $(a, b) \setminus N(f_0)$.

As a consequence of Proposition 1 we have that $\{1, g_1, \dots, g_n\}$ is a Markov system of analytic functions on (t_{j-1}, t_j) for every integer j. As in the proof of Proposition 1, no function in the linear span of $\{1, g_1, \dots, g_n\}$ has more than n sign changes on $(a, b) \setminus N(f_0)$, and so the set $\{1, g_1, \dots, g_n\}$ is a Markov system on $(a, b) \setminus N(f_0)$.

The functions $g_1, \dots, g_n$ may become unbounded in the neighborhoods of a and b, but not at any other points of the interval (a, b). Let us prove this statement by induction.

For $n \leq 1$, it is obviously true. Assume that all functions in the linear span of $\{1, g_1, \dots, g_{n-1}\}$ are bounded on every closed interval contained in (a, b). Suppose, furthermore, that g_n is unbounded at $c \in (a, b)$, where $c \in N(f_0)$ (observe that the only points at which g_n can be unbounded are in $N(f_0)$). Without loss of generality, assume that $g_n(t) \to \infty$ as t approaches c from the left. Since $N(f_0)$ is not dense in (a, b) we can find an interval (a', b'), such that $a < a' < b' < c$ and so that g_n is bounded on (a', b'). Then there exists a function g in the linear span of $\{1, g_1, \dots, g_{n-1}\}$ and points $(t_0, \dots, t_{n-1}) \in \Delta_n((a', d) \setminus N(f_0))$ with $d \in (a', b')$, so that $g(t_i) = (-1)^{n-i}$ for every $i = 0, \dots, n-1$. This implies that for a small $\varepsilon > 0$ and $f := (\varepsilon g_n + g) \in U_n$, we have sign $f(t_i) =$ sign $g(t_i)$ for every $i = 0, \dots, n-1$, and $f(t) \to \infty$ if $t \to c_-$. Therefore, there exist $t_n \in (d, c) \setminus N(f_0)$ with $f(t_n) > 0$ and a point $t_{n+1} \in (c, b) \setminus N(f_0)$ with $f(t_{n+1}) < f(t_n)$.

But then $(t_0, \dots, t_{n+1}) \in \Delta_{n+1}(I \setminus N(f_0))$ is an oscillation of length $n+2$ of f, in contradiction to the fact that $\{1, \dots, g_n\}$ is a normalized Markov system on $(a, b) \setminus N(f_0)$.

Since $f_0, \dots, f_n$ are analytic, for every $c \in N(f_0)$ there are natural numbers $k_i \in \mathbb{N}$, and analytic functions h_i with $h_i(c) \neq 0$, such that $f_i(x) = (x-c)^{k_i} \cdot h_i(x)$ for all x in a neighborhood B_c of c, containing no further zero of f_0, other than c. Therefore, we have for every $i = 1, \dots, n$

$$g_i(x) = (x-c)^{k_i - k_0} \cdot \frac{h_i(x)}{h_0(x)}.$$

Each h_i/h_0 is analytic on B_c, since $h_0(c) \neq 0$. All functions g_i are bounded, as x approaches c on the real line, we must have $k_i \geq k_0$. This proves that each g_i is analytic in B_c.

Therefore, the functions $\{1, g_1, \dots, g_n\}$ are extendable to a Markov system of analytic functions on the whole interval (a, b).

COROLLARY 2. *Every weak Markov system of analytic functions is normalizable with a representable Markov system of analytic functions.*

References

1. F. Schwenker, Integral representation of normalized weak markov systems, *J. Approx. Theory*, **68** (1992), 1–24.
2. R. A. Zalik, Nondegeneracy and integral representation weak markov systems, *J. Approx. Theory* **68** (1992), 25–34.
3. D. Zwick, Degeneracy in WT-Spaces, *J. Approx. Theory* **41** (1984), 100–113.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis